NEGLIGIBLE SETS & $\mathcal{R}$-INTEGRABLE FUNCTIONS

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FEB 2020

Definition 1. Let $A \subset \mathbb{R}$. Then A is called *negligible* (or a *set of measure zero*) if there exists a sequence of bounded-open intervals (I_n) such that

$$A \subset \bigcup_{n=1}^{\infty} I_n \text{ and } \sum_{n=1}^{\infty} |I_n| < \epsilon,$$

where $|I_n|$ denotes the length of the interval I_n.

Example 1. *$\emptyset$ is negligible*:

Choose $\epsilon > 0$. Consider the sequence of bounded-open intervals (I_n), where $I_n = (1, 1)$, $\forall\, n \in \mathbb{N}$. Clearly $\emptyset \subset \bigcup_{n=1}^{\infty} I_n$ and $\sum_{n=1}^{\infty} |I_n| = 0 < \epsilon$. Thus $\emptyset$ is negligible.

Example 2. *A subset of a negligible set is negligible*:

Let A be a negligible subset of $\mathbb{R}$ and $B \subset A$.

Choose $\epsilon > 0$. Then $\exists$ a sequence (I_n) of bounded-open intervals such that $A \subset \bigcup_{i=1}^{\infty} I_n$ and $\sum_{n=1}^{\infty} |I_n| < \epsilon$. Consequently, $B \subset \bigcup_{i=1}^{\infty} I_n$, where $\sum_{n=1}^{\infty} |I_n| < \epsilon$. Thus B is negligible.

Example 3. *Any finite subset of $\mathbb{R}$ is negligible*:

Let $A = \{x_1, x_2, \cdots, x_n\} \subset \mathbb{R}$.

Choose $\epsilon > 0$. Consider the sequence (I_n) of bounded-open intervals where

$$I_n = \begin{cases} \left(x_i - \frac{\epsilon}{4n}, x_i + \frac{\epsilon}{4n}\right), & 1 \le i \le n \\ (1,1), & i > n \end{cases}.$$

Then $A \subset \bigcup_{n=1}^{\infty} I_n$ and $\sum_{n=1}^{\infty} |I_n| = \frac{\epsilon}{2n} \times n < \epsilon$. Thus A is negligible.

Example 4. *Any countable subset of* $\mathbb{R}$ *is negligible*:

Let A be a countable subset of $\mathbb{R}$ given by $A = \{x_1, x_2, \cdots\}$.

Choose $\epsilon > 0$. Consider the sequence (I_n) of bounded-open intervals where

$$I_n = \left(x_n - \frac{\epsilon}{2^{n+2}}, x_n + \frac{\epsilon}{2^{n+2}}\right), \ \forall\, n \in \mathbb{N}.$$

Clearly $A \subset \bigcup_{n=1}^{\infty} I_n$ and $\sum_{n=1}^{\infty} |I_n| = \left(\frac{\epsilon}{2^2} + \frac{\epsilon}{2^3} + \frac{\epsilon}{2^4} + \cdots\right) = \frac{\epsilon}{2} < \epsilon$. Thus A is negligible.

Example 5. *Countable union of negligible sets is negligible*:

Let $\{A_n : n \in \mathbb{N}\}$ be a countable collection of negligible subsets of $\mathbb{R}$.

Choose $\epsilon > 0$. Then corresponding to each $i \in \mathbb{N}$, $\exists$ a sequence (I_n^i) of bounded-open intervals such that $A_i \subset \bigcup_{n=1}^{\infty} I_n^i$ and $\sum_{n=1}^{\infty} |I_n^i| < \frac{\epsilon}{2^{i+1}}$, $\forall\, i \in \mathbb{N}$.

Set (J_n) to be the sequence of bounded-open intervals given by $(I_1^1, I_1^2, I_2^1, I_1^3, I_2^2, I_3^1, \cdots)$. Clearly then

$$\bigcup_{i=1}^{\infty} A_i \subset \bigcup_{n=1}^{\infty} J_n \text{ and } \sum_{n=1}^{\infty} |J_n| = \sum_{i=1}^{\infty} \sum_{n=1}^{\infty} |I_n^i| \le \left(\frac{\epsilon}{2^2} + \frac{\epsilon}{2^3} + \frac{\epsilon}{2^4} + \cdots\right) = \frac{\epsilon}{2} < \epsilon.$$

Thus $\bigcup_{i=1}^{\infty} A_i$ is negligible.

Example 6. *Finite union of negligible sets is negligible*:

Let $A_1, A_2, \cdots, A_n$ be negligible subsets of $\mathbb{R}$ and $A = \bigcup_{i=1}^{n} A_i$.

Clearly $A = \bigcup_{i=1}^{\infty} B_i$, where

$$B_i = \begin{cases} A_i, & 1 \leq i \leq n \\ \emptyset, & \text{otherwise} \end{cases}.$$

Thus, in view of Example 5, A is negligible.

[For the direct proof, see *Mapa*]

Example 7. *A subset of* $\mathbb{R}$ *having countable derived set is negligible*:

Let $A \subset \mathbb{R}$ be such that $A^d = \{x_1, x_2, \cdots\}$.

Choose $\epsilon > 0$. Consider the sequence (I_n) of bounded-open intervals such that $I_n = \left(x_n - \frac{\epsilon}{2^{n+3}}, x_n + \frac{\epsilon}{2^{n+3}}\right)$, $\forall\, n \in \mathbb{N}$.

Let $B = \bigcup_{n=1}^{\infty} I_n \backslash A$. Clearly B is countable, $\{y_1, y_2, \cdots\}$, say.

Set $J_n = \left(y_n - \frac{\epsilon}{2^{n+3}}, y_n + \frac{\epsilon}{2^{n+3}}\right)$, $\forall\, n \in \mathbb{N}$.

Set (K_n) to be the sequence of bounded-open intervals given by

$$K_n = \begin{cases} I_{\frac{n+1}{2}}, & n = \text{odd} \\ J_{\frac{n}{2}}, & n = \text{even} \end{cases}.$$

Then $A \subset (\bigcup_{n=1}^{\infty} I_n) \bigcup B \subset \bigcup_{n=1}^{\infty} K_n$ and $\sum_{n=1}^{\infty} |K_n| = \sum_{n=1}^{\infty} |I_n| + \sum_{n=1}^{\infty} |J_n| = \sum_{n=1}^{\infty} \frac{\epsilon}{2^{n+2}} + \sum_{n=1}^{\infty} \frac{\epsilon}{2^{n+2}} = \frac{\epsilon}{2} < \epsilon$.

Thus A is negligible.

Example 8. *A subset of* $\mathbb{R}$ *having finite derived set is negligible*: Similar (simpler) – see *Mapa*, if stuck.

Notation 1. Given a function $f : [a, b] \to \mathbb{R}$, we denote the set of point of discontinuities of f on $[c, d]$ ($\subset [a, b]$) by $\mathcal{D}_f[c, d]$.

Theorem 1. *(Riemann-Lebesgue) Let $f : [a, b] \to \mathbb{R}$ be bounded on $[a, b]$. Then $f \in \mathcal{R}[a, b]$ if and only if $\mathcal{D}_f[a, b]$ is negligible.*

Corollary 1. *Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$. Then $f \in \mathcal{R}[a, b]$:*

Clearly $\mathcal{D}_f[a, b] = \emptyset$. We first show that $\emptyset$ is negligible: $(*)$.

Thus by the Riemann-Lebesgue theorem $f \in \mathcal{R}[a, b]$.

Corollary 2. *Let $f : [a, b] \to \mathbb{R}$ be bounded on $[a, b]$ such that f is continuous on $[a, b]$ except for a finite number of points. Then $f \in \mathcal{R}[a, b]$:*

Clearly $\mathcal{D}_f[a, b]$ is finite. We first show that every finite set is negligible: $(*)$.

Thus by the Riemann-Lebesgue theorem $f \in \mathcal{R}[a, b]$.

Corollary 3. *Let $f : [a, b] \to \mathbb{R}$ be monotone on $[a, b]$. Then $f \in \mathcal{R}[a, b]$:*

Since f is monotone on $[a, b]$, $\mathcal{D}_f[a, b]$ is countable. We first show that every countable set is negligible: $(*)$.

Thus by the Riemann-Lebesgue theorem $f \in \mathcal{R}[a, b]$.

Corollary 4. *Let $f : [a, b] \to \mathbb{R}$ be bounded on $[a, b]$ such that f is continuous on $[a, b]$ except for a countable number of points. Then $f \in \mathcal{R}[a, b]$* : SIMILAR.

Corollary 5. *Let $f : [a, b] \to \mathbb{R}$ be bounded on $[a, b]$ such that f is continuous on $[a, b]$ except for a subset A of $[a, b]$, where A^d is finite. Then $f \in \mathcal{R}[a, b]$* : SIMILAR.

Corollary 6. *Let $f : [a, b] \to \mathbb{R}$ be bounded on $[a, b]$ such that f is continuous on $[a, b]$ except for a subset A of $[a, b]$, where A^d is countable. Then $f \in \mathcal{R}[a, b]$* : SIMILAR.

Corollary 7. $f, g \in \mathcal{R}[a,b] \implies f+g \in \mathcal{R}[a,b]$:

Let $f, g \in \mathcal{R}[a,b]$. Then $\mathcal{D}_f[a,b], \mathcal{D}_g[a,b]$ are negligible.

We first show that negligiblity is closed under finite union: $(*)$.

Consequently $\mathcal{D}_f[a,b] \bigcup \mathcal{D}_g[a,b]$ is negligible. Since $\mathcal{D}_{f+g}[a,b] \subset \mathcal{D}_f[a,b] \bigcup \mathcal{D}_g[a,b]$, so $\mathcal{D}_{f+g}[a,b]$ is negligible.

Thus by the Riemann-Lebesgue theorem $f+g \in \mathcal{R}[a,b]$.

Corollary 8. $f \in \mathcal{R}[a,b] \implies kf \in \mathcal{R}[a,b],\ \forall\, k \in \mathbb{R}$: SIMILAR.

Corollary 9. $f, g \in \mathcal{R}[a,b] \implies fg \in \mathcal{R}[a,b]$: SIMILAR.

Corollary 10. $f \in \mathcal{R}[a,b] \implies |f| \in \mathcal{R}[a,b]$: SIMILAR.

Corollary 11. $f, g \in \mathcal{R}[a,b] \implies fg \in \mathcal{R}[a,b]$: SIMILAR.

Corollary 12. *$f, g \in \mathcal{R}[a,b]$ and $g(x) \geq k$ (for some $k > 0$) on $[a,b] \implies f/g \in \mathcal{R}[a,b]$:*

Since $g(x) \geq k > 0$ on $[a,b]$, so f/g exists and bounded on $[a,b]$. Since $\mathcal{D}_{1/g}[a,b] = \mathcal{D}_g[a,b]$ and $g \in \mathcal{R}[a,b]$, so $\mathcal{D}_{1/g}[a,b]$ is negligible.

Consequently, $1/g \in \mathcal{R}[a,b]$.

We now show that $u, v \in \mathcal{R}[a,b] \implies uv \in \mathcal{R}[a,b]$: $(*)$.

Since $f, 1/g \in \mathcal{R}[a,b]$, so by the Riemann-Lebesgue theorem $f/g \in \mathcal{R}[a,b]$.

Corollary 13. *Let $f : [a,b] \to \mathbb{R}, g : [c,d] \to \mathbb{R}$ be two maps such that $f[a,b] \subset [c,d]$. If $f \in \mathcal{R}[a,b]$ and $g \in \mathcal{C}[c,d]$ then $g \circ f \in \mathcal{R}[a,b]$:*

Clearly $g \circ f$ is defined on $[a,b]$.

Since $g \in \mathcal{C}[c,d]$, it follows that $\mathcal{D}_{g \circ f} = \mathcal{D}_f$. Also $f \in \mathcal{R}[a,b] \implies \mathcal{D}_f$ is negligible. Thus by the Riemann-Lebesgue theorem $g \circ f \in \mathcal{R}[a,b]$.

Corollary 14. $f \in \mathcal{R}[a,b] \iff f \in \mathcal{R}[a,c]$ *and* $f \in \mathcal{R}[c,b]$:

Let $f \in \mathcal{R}[a,b]$. Then $\mathcal{D}_f[a,b]$ is negligible.

We first show that negligiblity is closed under subsets: $(*)$.

Since $\mathcal{D}_f[a,c], \mathcal{D}_f[c,b] \subset \mathcal{D}_f[a,b]$, so $\mathcal{D}_f[a,c]$ and $\mathcal{D}_f[c,b]$ are negligible.

Thus by the Riemann-Lebesgue theorem $f \in \mathcal{R}[a,c]$ and $f \in \mathcal{R}[c,b]$.

Conversely, let $f \in \mathcal{R}[a,c]$ and $f \in \mathcal{R}[c,b]$.

We first show that union of two negligible sets is negligible: $(*)$.

Since $\mathcal{D}_f[a,b] = \mathcal{D}_f[a,c] \bigcup \mathcal{D}_f[c,b]$, so $\mathcal{D}_f[a,b]$ is negligible.

Thus by the Riemann-Lebesgue theorem $f \in \mathcal{R}[a,b]$.

Corollary 15. *Let* $f \in \mathcal{R}[a,b]$ *and* $g : [a,b] \to \mathbb{R}$ *be such that* $f(x) = g(x)$ *on* $[a,b]$ *except for finite (resp. countable) number of points. Then* $g \in \mathcal{R}[a,b]$: LEFT AS AN EXERCISE.

$(*)$: *Plug-in the respective proof*

Reference: REAL ANALYSIS BY S. K. MAPA